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DIDACTIC OBJECTS AND DIDACTIC MODELS IN RADICAL CONSTRUCTIVISM

This chapter discusses ways in which conceptual analyses of mathematical ideas from a radical constructivist perspective complement Realistic Mathematics Education's attention to emergent models, symbolization, and participation in classroom practices. The discussion draws on examples from research in quantitative reasoning, in which radical constructivism serves as a background theory. The function of a background theory is to constrain ways in which issues are conceived and types of explanations one gives, and to frame one's descriptions of what needs explaining.

The central claim of the chapter is that quantitative reasoning and realistic mathematics education provide complementary foci in both design of instruction and evaluation of it. A theory of quantitative reasoning enables one to describe mathematical understandings one hopes students will have, and ways students might express their understandings in action or communication. It is argued that conceptual analyses of mathematical ideas cannot be carried out abstractly. In contrast, it is found to be highly useful to imagine students thinking about *something* in discussions of it. In relation to this, the focus is on what one imagines to be the "something" teachers and students discuss, and on the nature of the discussions surrounding it. This type of conceptual analyses overlaps considerably with the Realistic Mathematics Education notion of emergent models in instructional design. There is, however, a difference in one respect; Realistic Mathematics Education attends to tools which will influence students' activity, while from a quantitative-reasoning perspective the focus is more on things students might re-perceive and things about which a teacher might hold fruitful discussions.

There is more than a little controversy as to whether radical constructivism (hereafter "constructivism") can contribute to theories of instructional design in mathematics education. Cobb, Gravemeijer, and colleagues (Cobb, in press; Gravemeijer, 1994b; Gravemeijer, Cobb, Bowers, & Whitenack, in press) suggest that constructivism has too strong an emphasis on individual activity and individual meaning to address issues that are more strongly highlighted by a focus on collective activity. Lerman (1994, 1996) considers the radical constructivist position to be barely coherent, and even if it is coherent in the way it addresses individual cognition, he claims it cannot address the issue of intersubjective agreement or

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shared knowledge, which is foundational to theories of instructional design. In this chapter I will first address the theoretical question of constructivism's capability to address these matters coherently. Then I will offer examples of how to do it in a way that complements approaches to instructional design having stronger connections to sociocultural perspectives on mathematical activity.

An oft-projected image of constructivism is that of an epistemological stance bordering nihilism or solipsism (Suchting, 1992), wherein individuals roam some terrain in complete isolation of one another, constructing worlds at their whim. However, as Glasersfeld (1990) states, "The constructivist is fully aware of the fact that an organism's conceptual constructions are not fancy-free. On the contrary, the process of constructing is constantly curbed and held in check by the constraints it runs into" (p. 33). It is important to make explicit that the "constraints it runs into" are not obstacles in a ready-made world, the one an observer isolates as that which is not the organism. Rather, the constraints an organism runs into are functional - obstacles the organism experiences in acting or knowing.

Moreover, people often forget that Glasersfeld's elaboration of Piaget's genetic epistemology into what he eventually called radical constructivism (Glasersfeld, 1978) grew in large part out of Glasersfeld's keen interest in understanding the nature of human communication and language (Glasersfeld, 1970, 1975, 1977, 1990; von Foerster, H., 1979). The core problems of radical constructivism, from its very beginning, entailed the question of how physically-disconnected, self-regulating organisms could influence each other to end in a state where each presumes there is common agreement on what is their shared environment (Maturana, 1978; Richards, 1991). That is, an image of cognizing individuals in social contexts was central to radical constructivism from its very outset.

Two tenets of constructivism are that we are biological organisms and that constructivism is reflexive. The first tenet implies that no one has unfettered access to his or her environment or to others' environments. The second tenet says that constructivism applies to everyone, including those who claim it applies to everyone. As biological organisms, we have access only to the residual effects of neurological activity, which include signals generated from within ourselves without immediate environmental contact (e.g., through imagination or reflection). The tenet that constructivism is reflexive is actually redundant with the first, but many people seem surprised to hear so (Phillips, 1996, 2000; Thompson, 2000).

Constructivism is a background theory, or, as Noddings (1991) said, a post-epistemological stance. Background theories cannot be used to explain phenomena or to prescribe actions. Rather, their function is to constrain the types of explanations we give, to frame our conceptions of what needs explaining, and to filter what may be taken as a legitimate problem. Constructivism provides a continual reminder that, regardless of how taken one becomes with his or her insights into how the world works, we are biological organisms whose only way to exert mutual influence, aside from physical harm or pleasure, is through mutual interpretation.

For constructivists, the constraints of constructivism as a background theory are always present—whether addressing the question of how individual students' come to understand a mathematical idea or addressing the question of how newcomers become initiated to what we take as cultural practices. Keeping this in mind helps us heed Simon's (1995) advice that we cannot propose anything like a "constructivist pedagogy." Constructivism says that whatever sense people make of their experience, they construct that sense themselves — *regardless of what anyone else does to influence it.* What a person actually ends up knowing can be influenced by what others do, but communication happens only through interpretation. This is why "telling" is not necessarily an anticonstructivist pedagogical action. There can be great pedagogical power in judicious use of direct instruction and great danger in an over-reliance on "discovery learning" approaches to teaching. It can be appropriate for teachers to describe explicitly to their students the understandings they hope their students will have. But after so describing those understandings, and after teaching for them, teachers should not presume that what students understand is what was intended. Instead, they should listen for cues as to what sense students have made of what was said or done, including asking for students' interpretations of it.

In summary, to pronounce constructivism as a background theory is not to announce a commitment to a particular theory of learning or to a particular type of pedagogy. Instead, it is to announce a set of commitments and constraints on the kinds of explanations one may accept and on the ways one frames problems and phenomena. A commitment to constructivism may have ramifications for the instructional actions one anticipates will be effective regarding students with certain characteristics coming to have particular understandings in the context of certain environments, but it does not, of itself, prescribe or exclude any particular action as being possibly effective.

While one's background theory and epistemology constrain and orient the kinds of explanations and descriptions one gives, an explanation's actual content comes from theories specific to what is being explained or described. Much of what I discuss in this paper derives from a theory of quantitative reasoning (Thompson, 1993, 1995, 1996), which itself relies heavily on Piaget's constructs of praxis (goal-directed action), schema, mental operation, and scheme. But I will not discuss that theory per se. Instead, I will draw from research in it to propose something that seems, on surface, an oxymoron—the idea of instructional design from a radical constructivist perspective. It is the issue of instructional design that brings together constructivism and Realistic Mathematics Education, at least that version practiced by Gravemeijer and Cobb (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer, 1994b; Gravemeijer et al., in press).

A rigorous development of a constructivist psychology and pedagogy would begin with biology (Maturana, 1978; Maturana & Verela, 1980; Powers, 1973; von Foerster, 1984) and consciousness (MacKay, 1965) and build toward mathematical thinking and communicating. For purposes of instructional design we needn't be so rigorous. We can be satisfied to devise descriptions of thinking and imagining that:

a) Fit with a constructivist framework,
b) Attempt to capture and communicate important aspects of students' and teachers' mathematical experience,

c) Propose ways of thinking, believing, imagining, and interacting that might be propitious for students' and teachers' mathematical development.

The previous paragraph might sound as if I changed epistemological canoes midstream. To propose "ways of thinking that are propitious for students' and teachers' mathematical growth" seems in conflict with "whatever sense a person makes of his personal experience, he constructs that sense himself — *regardless of what anyone else does.*" The apparent conflict is that, on one hand, the phrase "… constructs that sense himself …" often is heard as a form of nihilism or solipsism, which leaves no room for ideas like instructional design. The conflict is resolved by the idea of intersubjectivity. Before resolving the conflict, it may be worthwhile to note that "intersubjectivity" often is given two incompatible meanings. One meaning is that people reach a state of intersubjectivity when they hold identical meanings or share identical knowledge (Lerman, 1994, 1996). A second meaning of intersubjectivity is that people have reached a state of reciprocal assimilations where further assimilations are unproblematic (Steffe & Thompson, 2000; Thompson, 2000). It is the latter meaning that is consistent with constructivism. As Glasersfeld (1995) notes, to say two people communicate successfully means no more than that they have arrived at a point where their mutual interpretations, each expressed in action interpretable by the other, are compatible – they work for the time being. *Intersubjectivity* is a state where each participant in a socially-ongoing interaction feels assured that others involved in the interaction think pretty much as he or she imagines they do. That is, intersubjectivity is *not* a claim of identical thinking. Nor is it what Confrey (1991) described "agreeing to agree." Rather, it is a claim that each person sees no reason to believe others think differently than he or she presumes they do.

The idea of intersubjectivity is important to understanding instructional design from a constructivist perspective. One does not design instruction just with the idea that students engage in a particular activity or that they develop a particular interpretation. Rather, one designs instruction so that it will create a particular dynamical space, one that will be propitious for individual growth in some intended direction, but will also allow a variety of understandings that will fit with where individual students are at that moment in time. The design elements are objects of discussion and interpretations of them, with the realization that object and interpretation are defined reflexively. Instructional design from this perspective entails creating images of students and teachers that will reveal, to the extent possible, assumptions the designer is making about individuals participating in the discussions, and processes by which students might make significant mathematical advances (Thompson, 1985b). As such, one can take as a starting place images of students and teachers built from one's experiences as a teacher and researcher and from the understandings one builds of other persons' attempts at the same kind of work.[1]

I mean by "image" what Maturana describes as a conceptual system through which we may anticipate another system's behavior. In this regard, it is like an explanation.

> As scientists, we want to provide explanations for the phenomena we observe. That is, we want to propose conceptual or concrete systems that can be deemed intentionally isomorphic to the systems that generate the observed phenomena. (Maturana, 1978, p. 29)

It is important to note that Maturana did not stipulate a particular way in which a "system deemed intentionally isomorphic" might be expressed. It might be expressed in a computer program, in a system of equations, or in a verbal description of what happened and why it happened the way it did.

The images we build of individual students or teachers to explain their actions are images of *psychological* persons. Another sense of image, which is highly related to the one already given, is of an *epistemic* person (Piaget, 1971, 1977; Thompson & Saldanha, 2000). When imagining an epistemic person, we do not claim to be imagining anyone in particular. Rather, we speak generically of a person, as in "Suppose a person understands fractions as 'so many out of so many …" Images of this type allow us to propose ways of thinking that are not specific to any one person, yet they are not "disembodied cognizing". It is through the use of images (as described above) of an epistemic person that Tall (1989) can talk about the roles generic examples can play as long-term organizers of teachers' and students' activities. I suspect also that images of epistemic persons are highly related to what Cobb (in press) has in mind when he speaks of an envisioned practice as a goal of instructional design.

Quantitative Reasoning and Realistic Mathematics Education provide complementary foci in both design of instruction and evaluation of it. A theory of quantitative reasoning enables us to describe mathematical understandings we hope students will have and to describe understandings they do have, and how students might express their understandings in action or communication. Realistic Mathematics Education, with its emphasis on learning trajectories (Cobb, 1998; Simon, 1995), symbolization and tool use (Gravemeijer, 1994b; Gravemeijer et al., in press), and its recent emphasis on classroom mathematical practices as stable forms of participation (Cobb, in press), provides a powerful approach to instructional design. Its power derives from the natural fit it provides between theories of mathematical understanding and reasoning and the design of tasks, tools, and interventions from which those understandings might emerge in students' and teachers' participation in activities using them.

One of Realistic Mathematics Education's most appealing aspects is its attention to emergent models. As I understand it, emergent models are expressions of

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[1] An interesting question is to what extent this is a scientific enterprise.. I believe we benefit by *trying* to be scientific, but at the same time I believe we cannot, at this time, *be* scientific.

transitional, but stable, understandings and points of view the designers hope emerge through class discussion and activity. This emphasis puts squarely in center stage the question of what we hope students learn. The issue of what we hope students learn becomes explicit during the initial design of tools, tasks, contexts, and interpretations of them we hope emerge from students' activities. Theories of quantitative reasoning provide conceptual analyses of what we hope students eventually understand, and hence complement Realistic Mathematics Education's attention to emergent models, symbolization, and participation in social and sociomathematical classroom practices. Realistic Mathematics Education's explicit attention to didactic phenomenology as central to instructional design is highly compatible with Quantitative Reasoning's focus on the conceptual operations by which people conceive quantitative situations coherently.

CONCEPTUAL ANALYSIS AND DIDACTIC OBJECTS

It is useful to describe ways of knowing that operationalize what it is students might understand when they know a particular idea. Glasersfeld (1995) calls his method for doing this *conceptual analysis.* As Steffe (1996) notes, the main goal of conceptual analysis is to propose answers to the question, "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (Glasersfeld, 1995, p. 78). The method of conceptual analysis applied to mathematical ideas resembles Dubinsky's method of genetic decomposition (Dubinsky & Lewin, 1986). Dubinsky attempts to describe a scheme of interconnected mathematical ideas in a way that someone who understands them well might understand them. He does this with the anticipation of backtracking from descriptions of "well understood" ideas to describe transitional understandings someone might hold that can be expressed in actions and tensions which themselves support productive reflection.

Dubinksy's notion of genetic decomposition differs from conceptual analysis in one important respect. Conceptual analyses are given in terms grounded in conceptual experience – to make it clear that we're talking about *someone* having *something* in mind—that what they have in mind derives from and is constantly informed by imagery, emotion, intention, etc. (Thompson, 1996). While I find genetic decomposition to be a powerful method for thinking about what we want students to learn, without grounding it in conceptual experience it loses much of its power when attempting to describe how someone might learn it.

Conceptual analyses of mathematical ideas cannot be carried out abstractly. Instead, doing conceptual analyses entails imagining students thinking about *something* in the context of *discussing it*. As a personal note, even though conceptual analysis was foundational to my earliest work in instructional design, tool design, and curriculum design (Thompson, 1982, 1985b, 1985c), it is because of recent emphases on the social context of learning that I now realize it is important to say so. In that early work I described the growth of students' mathematical ideas in the context of their acting within computer-enhanced instructional environments. I did not say that these environments were designed always with the intention that they would provide occasions for a teacher to discuss “what” needed understanding and “how” to imagine it. I also did not say that these discussions were envisioned as providing opportunities for teachers to ensure that specific conceptual issues would arise. It is perhaps interesting to note that the working title of *Computers in research on mathematical problem solving* (Thompson, 1985a) was “What to do before pressing RETURN." The article was about the conversations an instructor (teacher or researcher) might promote by having students predict what will happen in a mathematical microworld upon pressing RETURN – all the time with a finger hovered above the RETURN key.[2]

It is in this respect, designing instruction with the intent of producing instructional conversations of a particular type and grounding that design in an analysis of understandings that would support them, that conceptual analysis overlaps with Gravemeijer’s and Cobb’s notion of emergent models in instructional design (Cobb, Gravemeijer et al., 1997; Gravemeijer, 1994b). As Gravemeijer et al. (2000) note,

> The term model as it used here can refer to a task setting or to a verbal description as well as to ways of symbolizing and notating. Thus, although we will speak of models and symbolizations interchangeably, there is a slight difference. In Realistic Mathematics Education, the term model is understood in a dynamic, holistic sense. As a consequence, the symbolizations that are embedded in the process of modeling and that constitute the model can change over time …. This approach of proactively supporting the emergence of taken-as-shared models involves both the judicious selection of instructional activities and the negotiation of the ways of symbolizing that students create as they participate in communal classroom practices. (pp. 240-241).

Thus, an emergent model in RME is something to be aimed for as an outcome of instruction in the same way as is an instructional conversation within conceptual analysis. At the same time, there is a difference between an emergent model and an intended instructional conversation. An emergent model in RME is intended to be something of which students become more or less aware and which has a tool-like character to them. An intended instructional conversation is something of which the teacher is aware, but students needn’t be aware of it in the same way or even aware of it at all. As such, when viewed as a method by which to anticipate the conceptual operations that underlie a particular way of thinking and therefore the design of conversations that might support students’ developing them, conceptual analysis can be viewed not just as a method of producing psychological models of individual's understanding, but as an important activity in instructional design.

The methods of designing instruction for producing instructional conversations and of emergent models differ in another respect. In Realistic Mathematics Education one attends to tools that will influence students’ activity. In the method of conceptual analysis, one thinks less of influencing students’ activity and more of

[2] I am dating myself here. These microworlds were written before the days of mice, menus, buttons, or click-and-drag interfaces.

describing things students might re-perceive and things about which a teacher might hold fruitful discussions with them. It is in this respect that we see the need to attend to what we take as the "something" we imagine teachers and students discussing and to the nature of the discussions we imagine surrounding it. Put another way, conceptual analysis supports the design of "things to talk about" that, were the larger objective held in mind by a teacher as he or she manages classroom discussions, could engender reflective discourse around a desired theme, issue, or way of thinking (Cobb, Boufi, McClain, & Whitenack, 1997; Thompson, Philipp, Thompson, & Boyd, 1994).

It often is useful to coin a phrase for an idea still germinating. So, I'll propose the phrase *didactic object* to refer to "a thing to talk about" that is designed with the intention of supporting reflective mathematical discourse. I hasten to point out that objects cannot be didactic in and of themselves. Rather, they are didactic because of the conversations that are enabled by someone having conceptualized them as such. In one sense, a didactic object is a tool, but one designed for a teacher who can conceive of ways he or she might use it to produce desirable student engagements and classroom conversations.

I give three examples of didactic objects in the remainder of this chapter and discuss what makes them didactic. The first example will be of the use of common diagrams as didactic objects. The second will show, I hope, that we cannot take for granted what is the conversation in which students actually participate. The third will illustrate using a psychological theory to inspire the design of didactic objects. Afterward I'll use these examples to discuss the transformation of didactic objects into didactic models and the relationship of this approach to design research in Realistic Mathematics Education.

Example 1 - Uncommon uses of common objects

Vergnaud (1983, 1994) developed the important distinction between additive and multiplicative reasoning. One difference between reasoning additively and reasoning multiplicatively is in the way students imagine comparisons between parts and wholes. When reasoning about an object additively a student focuses on parts and wholes without regard to relative size. When reasoning multiplicatively a student focuses on two quantities simultaneously, and relative size is foundational to his or her reasoning.

As a case in point, Hunting, Davis, & Pearn (1996) and Mack (1995) have noted the commonness of students reasoning additively about fractions when it is intended that they reason about them multiplicatively. Figure 1, which could be found in any school mathematics textbook that introduces fractions, is often provided within a discourse that emphasizes additive reasoning.

Figure 1.

A student who understands the collection in Figure 1 additively might think:

- 3 dark disks and 2 white disks
- 3 of 5 disks are dark
- there are 2 more disks than dark disks

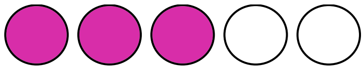
A student who understands Figure 1 multiplicatively might think:

- the number of disks is 5 times as large as one-third the number of dark disks (i.e., the number of disks is five-thirds the number of dark disks)
- the number of dark disks is 3 times as large as one-fifth the number of disks (i.e., the number of dark disks is three-fifths the number of disks)

If we presume that it is common for students to give additive meanings to fractional notation when we intend fractional notation to denote relative size, then it is an important instructional design question as to how we might support their changing from an additive conception of fractions to a multiplicative conception of fractions. In line with the aim to support students in that transformation, we can use Figure 1 as a didactic object to provoke a conversation wherein students who do ascribe additive meanings to fractional notation experience a conflict, and we can employ the conversation to help these students realize there is a discrepancy between what they see and what someone else sees.[3]

So, presuming we are working with students who have been introduced to fractions and many of whom reason additively about them, we might, as an opening to "a new view of fractions," hold a conversation around Figure 1as suggested in Figure 2.

[3] This conversation may not *fix* that problem. Fixing that problem will entail instruction aimed at their developing a scheme for multiplicative reasoning. A discussion of that instruction is beyond this paper's scope.



- Raise your hand when you can see 3/5 of something.
  (What do you see? What is the something of which you see three-fifths?)
- Raise your hand when you can see 5/3 of something.
  (What do you see? What is the something of which you see five-thirds?)
- Raise your hand when you can see 3/5 of 5/3.
  (What do you see? How much is three-fifths of five-thirds?)
- Raise your hand when you can see 5/3 of 3/5.
  (What do you see? How much is five-thirds of three-fifths?)
- Can you see 1 ÷ 3/5?
  (Recall that one meaning of "1 ÷ 3/5" is "The number of three-fifths in one.")
- Can you see 3/4 ÷ 5/4?
- Can you see 5/4 ÷ 3/4?

Figure 2.Directives and questions for Figure 1.

The discussion with students centers around a projected image of Figure 1; the directives ("Raise your hand …") appear one at a time under the projected image. A pause of suitable length follows each directive. The immediately ensuing discussion might be initiated by the parenthetical question (if many hands go up) or it might be about why it is hard to see what is indicated (if few hands go up).

The initial directives to students ("Raise your hand …") are intentionally *not* questions. Were they stated as, for example, "Can you see 5/3 of something?", students would answer *yes* or *no.* One effect of having students raise their hands is very practical. It helps convey the message that the activity's intent is that they reflect on something, not that they give an answer. Another practical effect is that lack of hands going up provides a natural occasion to ask "Why is it hard to see (whatever they are asked to see)?"

There is nothing special about the diagram in Figure 1. It could be found in any elementary text that addresses fractions. Its being in a textbook, though, does not make it the didactic object depicted in Figure 2. It becomes that didactic object in the hands of someone having in mind a set of images, issues, meanings, or connections affiliated with it that focus on interpreting it multiplicatively and which the teacher realizes must be discussed explicitly. Put another way, it is Figure 2 that depicts the didactic object of this example, not Figure 1. Without making clear the conversation intended to go with Figure 1, you have a merely a picture, not a didactic object.

Sowder and Philipp (1995) discussed the ramifications of this distinction—between standard interpretations of a diagram and a discussion tailored around it to move people's thinking in a particular direction. They describe a seminar in which a group of practicing teachers grappled with issues that emerged while discussing Figure 2, noting a dramatic shift in the level of discussion from "what do I see" to a discussion of the conceptual operations by which one gives meaning to fractions. Sowder and Philipp attributed this shift to the discussion's choreography, which points to the importance of designing "things to talk about" with an eye toward the conversations in which their use might support reflective discourse.

Example 2 — Didactic objects and shared meanings

In this example I share parts of a teaching experiment to investigate the roots of college mathematics seniors' difficulties understanding the Fundamental Theorem of Calculus (Thompson, 1994b). The purpose of including this example is to make explicit the problematic nature of assuming common meanings and interpretations among students even when explicit agreement apparently is reached on what something means.

One aspect of this teaching experiment uncovered students' dispositions to use notation unthinkingly.

> I should point out that the above discussion is colored by one serious matter. This is that students often acted from an orientation which led them to use notation opaquely. We discussed this tendency during class on several occasions. A common remark was that this seemed, from their point of view, the most efficient way to cope with what they thought had been expected of them, both in high school and in college. When students did interpret notation, it often came as an afterthought, and they often tended to read into the notation what they wanted it to say, without questioning how what they actually wrote might be interpreted by another person. More often, though, students would not interpret the notation with which they worked, but would instead associate patterns of action with various notational configurations and then respond according to internalized patterns of action. Their orientation toward notational opacity, while having nothing to do with conceptual difficulties with the Fundamental Theorem of Calculus as such, certainly contributed to their not having grappled with key connections. (p. 254)

Example 2 draws from one activity in which I forced an issue of how to formulate and interpret a rate of change in terms of a quotient of changes. As background to this example, I had designed the course so that, for several weeks prior to the session in which the present example is drawn, the course had aimed to have students rebuild their understandings of average rate of change. To raise the issues I foresaw as central to making the Fundamental Theorem a discussible idea, students needed to realize that constructing a coherent interpretation of formulas and graphs was more problematic than they thought.

The class session I describe here lasted approximately 90 minutes. It centered around interpreting a formula that defined an "average rate of change" function that itself was defined in terms of a function that expressed a square's area in relation to the square's side length (Figure 3). I will describe this session in some detail.

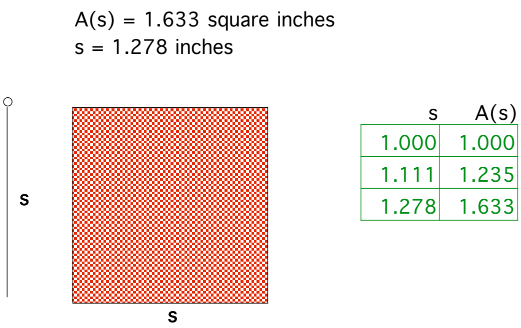


Figure 3. Square region whose side length and area varies. Table shows various side lengths and associated areas.

A theme of the course up to this session had been to develop a consensual understanding of average rate of change of one quantity (A) with respect to changes in another (B). The "consensual meaning" of average rate of change by this time was "the constant rate of change of A with respect to B that will produce the same change in A as actually happened in conjunction with the actual change in B".

The instructional segment began with students watching the square in Figure 3, displayed in Geometer's Sketchpad, as I increased the length of each side of the square by dragging the endpoint of the vertical segment labeled *s*. After each change in the side length I asked, "What was the average rate of change of the square's area with respect to the change in its side length?" For each computed value (e.g., .235/.111 for the first entry) I asked, "What does it mean that the average rate of change of area with respect to side length was 2.117 in this step?" In this case we proceeded after establishing that it meant *if the area were to increase at the constant rate of 2.117 in^2 per inch increase in side length as the square's side length increases from 1 inch to 1.111 inches, then the area would increase precisely as much as it actually did*. We repeated this discussion for several more values of *s* and A(*s*).

We observed that the side of a square, measured in inches, is increasing in length and that the square adjusts appropriately as the side's length increases. The area of the square, in square inches, was given at all times by $A(s) = s^2$. We also established that each value of the function

$$r(s) = \frac{A(s + 0.25) - A(s)}{0.25}$$

gave the average rate of change of area with respect to change in side length as the side-length changes from *s* inches to *s*+0.25 inches. This means that for any side-length *s*, were the area to increase for the next 0.25 inch increase in side length at the constant rate of change given by *r*(*s*), then the total change in the constantly-changing area would be precisely the same as the total change in *A*(*s*) over the interval [*s*,*s*+.25].[4]

I displayed the graphs of *A*(*s*) and *r*(*s*) (see Figure 4) and said "Here is a point on the graph of *r* [clicking on a point of *r*'s graph, displaying coordinates (.7667,1.635)]. What does this point, the one having coordinates (0.7667,1.635), represent?"

Students formed groups of two or three, each group being directed to arrive at a consensus statement of what is represented by the point (0.7667,1.635). After all were finished, each group reported the result of their discussion. Everyone in every group stated his or her satisfaction that their spokesperson represented their group's interpretation, and everyone stated his or her satisfaction that the groups arrived at essentially identical interpretations. I then asked each person to write, individually, his or her answer to the question, "What is represented by the point on *r*'s graph having coordinates (0.7667,1.635)?" Students were surprised by my request. Several stated, "We already said what it means." I explained my request's purpose—to see what, individually, they had gotten from the public discussion.

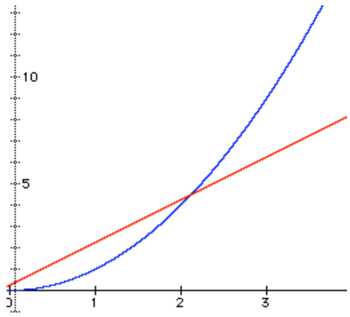


Figure 4. Graphs of y=A(s) and y= r(s) in one coordinate system.

Eleven of 19 students said what the coordinates represented. The remaining eight said they couldn't remember what their group had said before, that they couldn't reconstruct it, or they couldn't answer the question. Responses from the 11 students are given in Table 1.

[4] I fully appreciate anyone's right to wonder what it means that "we established" these interpretations. I employed the criterion that several people offered this interpretation, explained it, and no one objected that the speakers' interpretation or explanation was problematic (Yackel & Cobb, 1996).

Table 1. Responses to "What is represented by the point (0.7667,1.635)?"

1. That point represents the average rate of change in the area with respect to the change in length of the side at .7667 inches. Ave. rate of change $= \frac{1.635 \text{ sq in}}{\Delta \text{ in s in inches}}$	2. Given side length *s* of .7667 in., the avg rate of change in area with respect to side length is 1.635 sq/in.
3. *c*=.7667 is the rate change of side length at this point. *y*=1.635 is the average rate of change to respect of change of *x*.	4. r(x) represents avg rate of change. Therefore, the specific point represents rate of change from area of sq with side *s* to (*s*+.25) where side *L* = .7667. Avg. rate of change here is 1.6.
5. *x*=.7667, *y*=1.635. At point *x* = .7667, if the rate of change of r were to be constant thereafter, the area would increase by 1.635 every time x increased by 1.	6. What does the point *x*=.7667 and *f*(*x*) = 1.635 mean? The average rate of change in the area in sq in of a square with side .7667 in as it is increased by .25 in.
7. The point represents the area of the square with a side of .7667 using the average rate of change for a change of .25 inches in the side.	8. At *s* = .76 we find the average rate of change between a square of length .76 and a square of length approximately 1.
9. *r*(*s*) represents the rate of change of the area when the side changes from about 0.75 to 1.00.	10. What does the point *s*,r(*s*) stand for when *s* = .75?. $\frac{r(.75+.25)-r(.75)}{.25}$ $\frac{\text{rate of change in area}}{\text{change in s}}$.
11. What does (*x*,*y*) represent? An average rate of change of 1.635 units2 for side length of .7667 units increased by .25 inches.	

The responses in Table 1 are striking. First, none of them is internally consistent. Second, each response differs in important ways from the consensus statement arrived at on several occasions during group and class discussions. Response 1 is incomplete at first, and then gives an inappropriate formula, confounding values of $A(s)$ with changes in $A(s)$. Response 2 also confounds the ideas of area and change in area. Response 3 confuses rate of change of area with change in the side length. Response 4 is incomplete in comparison to the public statement in that it does not say what "average rate of change" actually means. Response 5 is actually quite creative. Had it said "… rate of change of *area*" instead of "rate of change of *r*" it would have been correct. The remaining responses have similar shortcomings.

The two aspects together, lack of internal coherence in their interpretations and lack of agreement between private and publicly stated interpretations, points to a matter worth considering. When we claim that agreement has been reached on a relatively complex idea because no disagreement has been expressed (as proposed by Yackel & Cobb, 1996) we must consider the possibility that students haven't analyzed their own or others expressions sufficiently to detect severe inconsistencies. Therefore, when employing didactic objects to ground conceptually-oriented discussions, it is prudent to exercise caution before concluding that a consensus in meaning or interpretation has been reached.[5] It may also be prudent to

[5] On the other hand, in everyday instructional settings it may be practical to pretend agreement has been reached knowing at the same time that it probably hasn't.

be cautious about concluding what individuals understand even when public agreement seems certain. What individuals understand might enable their participating in acceptable stable patterns of interaction, but the extent to which interaction-as-stable-pattern reflects acceptable states among individuals' understandings is uncertain at best.

This example follows up on the previous one (graphs as emergent aspects of covariation) in that it shows how an instructor might take a standard item of common instruction, a difference quotient, and use it as a didactic object to focus a discussion on a somewhat novel setting. It also raises an additional point: That we must be cautious about understandings we impute to individuals even when we take great care to ensure public consensus of interpretation and meaning.

Example 3 - Proffered images as didactic objects

One outcome of the teaching experiment from which Example 2 is drawn was that it showed how persistent were students' images of graphs as unanalyzed objects. The graphs they imagined were like solid pieces of wire, an image unsupportive of thinking about graphs as composed of points each of which represents a simultaneous state of two variable quantities at some moment in their covariation. The second image, graphs as emerging from having kept track of two quantities' values simultaneously as they varied, became the focus of another sequence of activities in a later course. The design of these activities draws heavily upon Piaget's constructs of internalization and interiorization, and are highly compatible with Johnson's (1987) ideas of bodily-centered meaning and metaphoric projection and Freudenthal's (1983) idea of didactic phenomenology. The objective is that students develop an operative image of covarying quantities whose simultaneous states at any moment are recorded in a point's coordinates.

Before entering this example it may be productive to examine some conceptual difficulties that obstruct students' understanding of a graph as presenting a record of two quantities' values simultaneous variation.

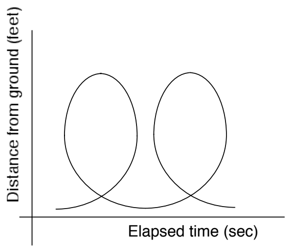


Figure 5. Can this graph show a Ferris wheel car's height at each moment during two revolutions?

The question stated about Figure 5 is whether that particular graph can be understood to depict a Ferris wheel car's height above ground as the elapsed time during which it made two revolutions varied. Students who answer "No, because it shows the Ferris wheel car going back in time" appear to be following the graph's physical contour, an action that gives them the sense of moving forward and backward along the *time* axis. Put another way, they imagine the graph as "where the action is," letting their eyes follow the graph's contour and noticing that they glance left-to-right above the *time* axis (Figure 6a). This is different from thinking first of elapsed time and the car's height varying in conjunction with each other.

If, on the other hand, students' attention is on the increasing amount of time the Ferris wheel has spun in conjunction with attending to the Ferris car's height at each moment in time, then the graph does not show "going back in time." Instead, it shows the Ferris wheel's car being at several different heights simultaneously (Figure 6b). In fact, when focusing on the covariation of elapsed time and car's height, the idea of "going back in time" is nonsensical. Time marches on (the clock's hand moves inexorably) and the Ferris car moves up and down. The graph, as a collection of points, merely captures their simultaneous states.

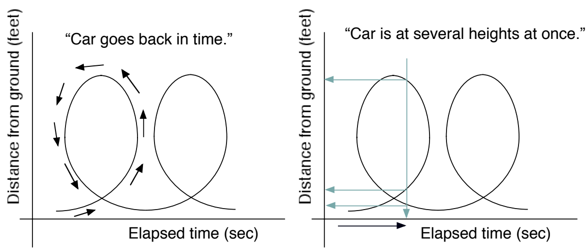


Figure 6. Two understandings of what Figure 5 shows.

It would require a separate paper to explore the negative ramifications of students seeing graphs only as trajectories, or worse, as if pieces of wire (Bell, 1981; Clement, 1989; Dugdale, 1993; McDermott, 1987) and to explore the positive ramifications of their seeing graphs as if emerging from someone having tracked the simultaneous variation of two quantities' values (Chazan, 1993; Confrey, 1995; Confrey & Smith, 1995; diSessa, 1991; Dugdale, 1992; Goldenberg, 1992; Smith, 1992; Tall, 1988; Thompson, 1994d). For the present purpose, I must ask you to assume that the "covariation" conception is desirable and worthy of having instruction designed for it.

The goal that students conceive graphs as records of simultaneous variation implies an early focus on having them internalize their perceptions of two quantities whose values vary, making that variation experientially concrete. To make covariation of quantities values experientially concrete, it is essential that they envision a single quantity's variation as itself having momentary states and therefore that the attribute whose value varies has momentary values. Likewise, if students are to see a graph as composed of points whose locations represent momentary states of two quantities' values simultaneously, then they need to think of lines and curves as being composed of points.

This all composes a thematic focus on the idea that, speaking metaphorically, a graph is an ephemeral byproduct of someone having monitored quantities' covariation and having used points' locations to capture momentary states simultaneously. The instructional design issue is to create didactic objects that will support the teachers' efforts to create instructional conversations that focus on these matters, happening eventually at a level of reflective discourse. One can use many items as a didactic object to support instruction toward that goal. The only essential feature is that students find in it two quantities whose values vary simultaneously and understand the task as to devise a method that records all simultaneous momentary states.

One instructional sequence[6] that entails the thematic focus described above builds on an analysis of students' construction of speed as a quantification of motion (Thompson, 1994a). The instructional sequence described here was used with 5th-graders and 8th-graders.[7] The opening phase concentrates on having students re-conceive the drawing of a segment as actually creating points. They tend to think that drawing a segment does not create any points; to create a point requires more of a dotting motion. The teacher asks them to draw a line segment, and then asks if they have drawn a point. Most students answer "no," although a few occasionally will already have a strong conception of segments as being composed of points. The teacher then hands out magnifying glasses so that students can examine the lines they have drawn, and generally they see that, indeed, what appears to the unaided eye as a solid line actually is made of spots of ink or pencil lead. This segues into a discussion of whether they could ever make a solid line that didn't reveal "spots" at some magnification (with "no" as the anticipated answer).[8]

Students are first asked to make a point, then 20 points, then 1000 points along a line. Someone usually offers at the "1000 points" phase that you can just draw a segment, but not everyone sees this connection. This brings out opportunities to discuss the reversibility between thinking of points comprising a line and a line being composed of points.

The group then watches a computer simulation of a rabbit running both ways ("over" and "back") along a linear track (Figure 7). In this case, the rabbit travels across the track at 40 ft/sec and back at 70 ft/sec. The group watches it several times, talking about what it means to run at 40 (70) feet per second. The teacher then pauses the simulation at various moments so that the group can talk about how far

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[6] I hesitate to use the phrase "instructional sequence" because it suggests a fixed order in which events happen. What actually happens is an unfolding of conversational phases rather than a fixed order of events. So, the actual conversations vary from implementation to implementation, but they are more or less similar to what I describe here in the phases that unfold.

[7] It has been used in slightly different forms with college elementary education majors, senior mathematics majors, and Ph.D. mathematics education students.

[8] Only the very advanced students raise the issue of continuity as being "no spots at any magnification," which then turns into a conversation about physical representation versus conception.

the rabbit had run in the number of seconds that had elapsed when it was paused. The simulation displays the amount of time and the distance the rabbit had run each time "pause" was pressed. Discussions of the projected display both clarify their image of what is being displayed and allow the teacher to highlight surreptitiously that at each moment during the rabbit's trip it had both traveled a specific number of feet and had used a specific number of seconds to do it.

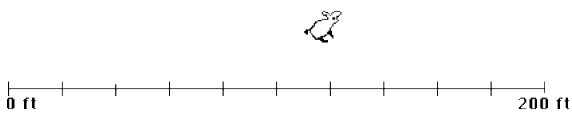


Figure 7. The rabbit in the midst of its trip.

The sequence progresses to a phase where the teacher asks students to use one finger to sweep out an imaginary segment whose length represents the amount of time the rabbit had run, and to do that as the rabbit runs (Figure 8). In a subsequent activity, students use their other hand to sweep out imaginary segments with their finger tip to keep track of the distance the rabbit has traveled (Figure 9). After practicing the actions depicted in Figure 8 and Figure 9 the teacher then asks them to do the two activities simultaneously, but asking only that their "swept distances" be from the same starting point (Figure 10). Finally, the teacher asks them to keep their distance finger directly above their time finger (Figure 11a). The group then talks about how many of them know about Tinkerbell, the pixie who flies around Peter Pan's Neverland leaving trails of pixie dust wherever she goes (even now most children, even immigrants, know about Tinkerbell). The group is told that unbeknownst to them the teacher has placed invisible bowls of pixie dust on their desks, and ask them to dip their "distance" finger into it. The children then re-enact their tracking of distance and time simultaneously as in Figure 11a, and they imagine their finger leaving a trail of pixie dust (Figure 11b). The group talks about whether the trail is solid (no, it is made of pixie dust particles) and about what each particle of pixie dust represents (a number of seconds the rabbit had run and the number of feet it ran in that number of seconds) and how to see these numbers (as marking distances from a starting point).

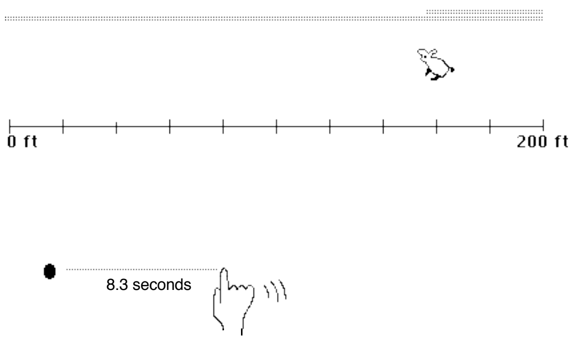


Figure 8. A student's hand as she "sweeps" a duration.

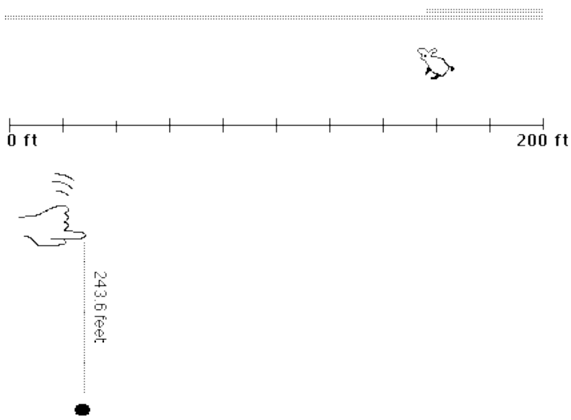


Figure 9. A students hand as she "sweeps" a total distance.

The pertinent point of this activity is that it takes a dynamic phenomenon as the object to which students attend and about which the teacher attempts to shape the public conversation. The activity is about using kinesthetic action representationally, not about just something seen. Students transform concrete actions, in their conceptions, into a model of something in the phenomenon that they hadn't seen.

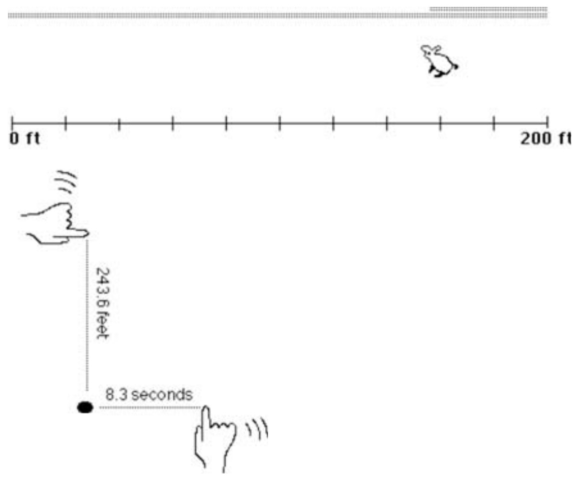
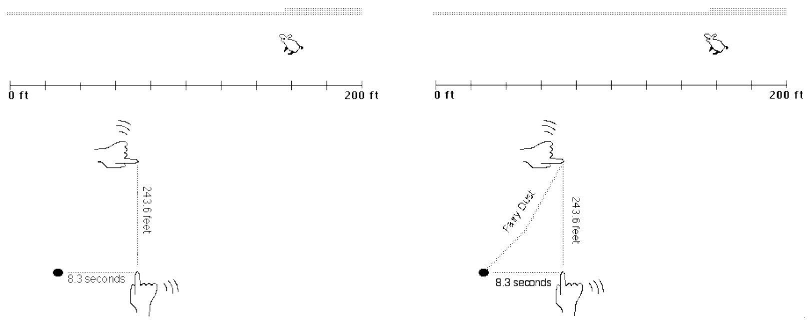


Figure 10. Keeping track of the two quantities simultaneously.



(a) (b)

Figure 11. Graphs as emergent objects, arising from someone having kept track of covariation.

The design of this activity draws heavily on the idea of having students engage in concrete action in a way that provides opportunities for teachers and students to engage in discourse that supports reflective abstraction of mathematical structure (Thompson, 1985b).[9] As already mentioned, this approach to instructional design is highly compatible with Freudenthal's notion of didactic phenomenology

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[9] Saldanha and Thompson (1998) extend this approach into the phases *engagement*, *move to representation*, and *move to reflection*.

(Freudenthal, 1983), which itself is foundational to instructional design in Realistic Mathematics Education (Gravemeijer, 1994a, 1994b). It differs somewhat from didactic phenomenology in its theoretical connections, though, in that an approach employing didactic objects and models is aimed explicitly at fostering abstraction of mental operations and operative mathematical structures, and those processes happen largely in the context of tasks designed to support conversations about the objects. So, it is not concrete activity per se that is the salient feature of instruction that employs didactic objects. Rather, the salient feature of this approach is the use of concrete activity or images thereof in conjunction with public conversations of it that are designed to promote individual's reflection on that activity in a way that leads to their abstracting mathematical structure from it.

Didactic Models

The phrase *didactic object* refers to "a thing to talk about" that is designed to support reflective mathematical discourse involving specific mathematical ideas or ways of thinking. It is worth repeating that objects are not didactic in and of themselves. Rather, they are didactic because of the conversations that are enabled by someone having conceptualized them as such. A natural question, from a design perspective, is whether we can assist others in creating didactic objects.

I mentioned in Example 3, the instructional sequence in Appendix I designed to promote students' internalization of actions that would lead to them conceiving graphs as records of simultaneous variation, that students transform concrete actions into models of phenomena that enabled them to see aspects they hadn't seen before. In that example, what they hadn't seen before was that at every moment during the turtle's motion it had traveled some total distance, and that this relationship between elapsed time and distance traveled itself revealed itself in a structure, an emergent pattern. There were many aspects of the pattern that they "knew" prior to the activity, such as that the rabbit ran at two different speeds. They also knew that constant speed gave "straight" graphs. But they couldn't explain why constant speed *should* give straight graphs. Rather, they knew that constant speed and straight graphs were somehow associated.[10] In other words, there were many aspects of speed and graphs that the students knew in isolation of each other. Rabbit's motion and their activity of tracking elapsed time and distance traveled, along with the issues that emerged because of the instructor's (i.e., my) agenda, brought those aspects into close proximity for them. This does not mean that we should expect students to immediately make deep and long-lasting connections among new, existing, and newly transformed conceptions. That takes prolonged reflection and practice. But it means we can hold reasonable expectations that students engage in

[10] It is important to note that in this approach to having students internalize graphs as records of covariation, that whether a graph is linear or not depends only on the phenomena being modeled, not on the activity of modeling it.

prolonged reflection and practice.

Other aspects of speed can be brought out in activities surrounding Rabbit's motion. Here is a partial list:

- the intrinsic proportionality of constant speed and how it is reflected in graphs as records of simultaneous variation;
- the difference between constant speed during a trip (which is constant over some interval of time) and average speed over any part of the entire trip (which is not constant over intervals of time during which Rabbit travels at different speeds), and how to see this distinction in a graph;[11]
- Rabbit's speed being constant, but different, over many small segments of his trip over and back and how that would be reflected in a graph;
- developing a formula which will give the distance Rabbit will have traveled at any moment during his trip;
- continuous changes in distance in relation to elapsed time but discontinuous changes in rate of change of distance and how that is revealed in graphs

This partial list shows a host of issues that can be brought out in the context of analyzing Rabbit's motion and the conceptual activity of tracking the quantities of elapsed time and distance traveled simultaneously. How might an instructor decide what issues to place on his or her agenda and when is an appropriate time to activate them? Clearly, the instructor needs a framework for thinking about the purpose and aims of the didactic object.

Here is where the notion of didactic model becomes useful. By *didactic model* I mean a scheme of meanings, actions, and interpretations that constitute the instructor's or instructional designer's image of all that needs to be understood for someone to make sense of the didactic object in the way he or she intends. But the model is of little use instructionally if it captures only sophisticated, advanced, coherent understandings. It is of greater use instructionally if it also captures aspects of how someone's understandings might evolve into sophisticated, advanced, coherent understandings. If a didactic model does capture aspects of how someone might develop advanced understandings of didactic objects, then it will of necessity address instructional actions someone might take to foster that development. Why? Because didactic objects entail images of conversations an instructor might have with students about the objects of discussion, or that students might have among themselves, collectively or individually, that will be propitious for the kinds of engagement out of which advanced understandings might emerge.

It is important to note that a didactic model is not a model for students of something they have experienced. They will create models, but those will not be didactic models. Rather, a didactic model is for instructors and instructional designers of what they intend students will understand and how that understanding might develop. Didactic models, as described here, are highly compatible with Simon's (1995) idea of a learning trajectory as a path by which understandings

[11] See the discussion of average rate of change in Example 2.

might develop, especially as it is employed by Gravemeijer and Cobb in their design research methodology (Gravemeijer, 1994b; Gravemeijer et al., 2000). One major difference between the idea of didactic model as used here and the idea of learning trajectory as used by Simon, Gravemeijer, and Cobb is didactic models' clear separation between descriptions of instructional sequences and descriptions of what students are to understand. Instruction and learning are less clearly separated in the idea of learning trajectory as employed in design research and Realistic Mathematics Education.

Didactic models' clear separation between descriptions of instruction and descriptions of learning provides advantages and disadvantages over the idea of a learning trajectory. One advantage is that it provides for the possibility of multiple approaches to the same goal. It allows for thinking of the two (learning and instruction) in relation to each other, yet separately. The learning objective cannot be confounded with the instructional sequence. It makes clear the possibility that successful implementation of an instructional sequence might lead to students' learning something very different from what was intended. One disadvantage is that it is *very* hard to develop learning objectives that will support a didactic model and make explicit the purposes of didactic objects on which it is based. It is almost as hard to communicate them.

The advantage of the idea of learning trajectories is that users of them needn't have a background in a psychological theory. They *can* confound means and ends productively. To borrow a phrase from Nemerovsky (this volume), they can fuse the instructional sequence and its intended outcomes. This fusion actually may make it easier for people to implement the instructional sequence. But this fusion may make it harder for an instructor or instructional designer to either recognize or deal with unintended outcomes (Thompson & Thompson, 1994, 1996).

I've offered the ideas of didactic objects and didactic models as a way to think of organizing the activity of instructional design. I believe they offer a way to rethink instructional design so that is consistent with a radical constructivist perspective while making explicit connections between instruction and learning.

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